Separated morphisms (Har II 4, Shaf V 4.3)

Let X be a Y-scheme. The diagonal morphism is the unique morphism $\Delta : X \longrightarrow X \times_Y X$ whose composition w/ projections $X \times_Y X \to X$ is the identity $X \to X$.

$$
\begin{tikzcd}
X \arrow[dr, dashed, "\Delta"] \arrow[drr, bend left, "id"] \arrow[ddr, bend right, "id"'] & & \\
& X \times_Y X \arrow[r] \arrow[d] & X \arrow[d] \\
& X \arrow[r] & Y
\end{tikzcd}
$$

Recall that a topological space is Hausdorff iff the image of the diagonal map $(id, id) : X \to X \times X$ is closed. However the underlying topological space of a scheme is almost never Hausdorff (many points are not even closed!). The analogue of the Hausdorff property for schemes is separatedness.

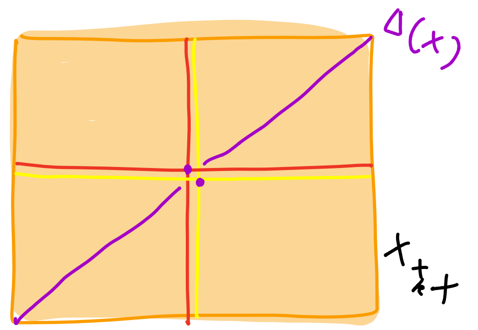
Def: A morphism $f : X \to Y$ is separated if the diagonal $\Delta : X \longrightarrow X \times_Y X$ is a closed immersion. In this case, we say X is separated over Y.

Remark: If X is a scheme of finite type over $\mathbb{C}$, we can give X the structure of a complex analytic space, and X is separated over $\operatorname{Spec} \mathbb{C}$ $\Longleftrightarrow$ the analytic topology on X is Hausdorff.

For any ring R, there is a unique homomorphism $\mathbb{Z} \to R$. $\mathbb{Z}$ is called an <u>initial object</u> in the category of rings. Thus, there is a unique morphism $\operatorname{Spec} R \to \operatorname{Spec} \mathbb{Z}$. Since every scheme X is covered by open affines, any morphism $X \to \operatorname{Spec} \mathbb{Z}$ must be unique. In fact, there is always such a morphism (Exercise).

Def: A scheme X is <u>separated</u> if it is separated over $\mathbb{Z}$.

Ex: let $\overset{= U_1 \cup U_2}{X}$ be the affine line over k with doubled origin. The scheme $X \times_k X$ is covered by four open affines, each isomorphic to $\underset{U_i}{\mathbb{A}^1_k} \times_k \underset{U_j}{\mathbb{A}^1_k} \cong \underset{V_{ij}}{\mathbb{A}^2_k}$, and is roughly $\mathbb{A}^2_k$ w/ doubled axes and four origins. The image of X intersected w/ V_{12} has closed points (x, x) s.t. $x \in U_1 \cap U_2$, (*) which is the line $y = x$ minus the origin, which is not closed. Thus X is not separated over k.



(*) convince yourself this is true

Prop: Any morphism of affine schemes $f: \operatorname{Spec} A \to \operatorname{Spec} B$ is separated.

Pf: $\operatorname{Spec} A \times_{\operatorname{Spec} B} \operatorname{Spec} A \cong \operatorname{Spec}(A \otimes_B A)$, and the diagonal morphism corresponds to the ring map

$$A \otimes_B A \longrightarrow A$$
$$a_1 \otimes a_2 \longmapsto a_1 a_2$$

which is surjective. Thus Δ is a closed immersion.

Cor: A morphism $f: X \to Y$ is separated $\Leftrightarrow$ $\Delta(X) \subseteq X \times_Y X$ is closed.

Pf: ($\Rightarrow$) follows from def of closed immersion.

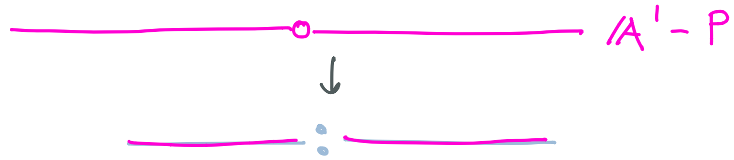
($\Leftarrow$) Need to show: $\Delta: X \to \Delta(X)$ is a homeomorphism and $\mathcal{O}_{X \times_Y X} \to \Delta_* \mathcal{O}_X$ is surjective.

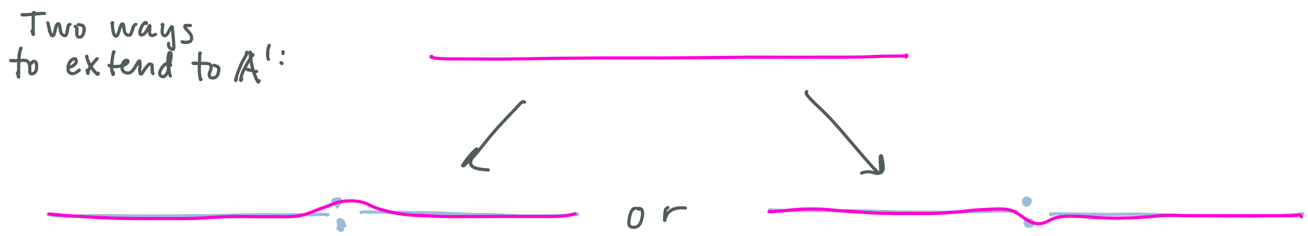
Since the composition $X \xrightarrow{\Delta} X \times_Y X \xrightarrow{p_1} X$ is the identity, Δ must be a homeomorphism.

Now take $P \in X$ and $U \subseteq X$ a nhood of P small enough so that $f(U) \subseteq V$, an open affine in Y. Then $U \times_V U$ is an open affine neighborhood of $\Delta(P)$, and $\Delta: U \to U \times_V U$ must be a closed immersion by the prop. Thus, the map of sheaves $\mathcal{O}_{X \times_Y X} \longrightarrow \Delta_* \mathcal{O}_X$ is surjective in a neighborhood of P, so it's surjective. (Do you see why this is sufficient?). $\square$

Valuative criterion of separatedness

Roughly, X is separated if for any curve C and $P \in C$, given a morphism $C - \{P\} \to X$, there is at most one way to extend the map to all of C. Of course $\mathbb{A}^1$ w/ doubled origin fails:





This is a local condition, so we can give a criterion on the stalk of C at P, which is a DVR (or more generally a valuation ring). We first review some algebra:

Def: Let K be a field and G a totally ordered abelian group (usually $\mathbb{Z}$). A <u>valuation</u> of K w/ values in G is a map $v : K - \{0\} \to G$ s.t.

- $v(xy) = v(x) + v(y)$, and
- $v(x+y) \geq \min(v(x), v(y))$.

The subring $R = \{x \in K \mid v(x) \geq 0\} \cup \{0\}$ is a <u>valuation ring</u> if it's an integral domain w/ field of fractions K.

R is local w/ max'l ideal $m = \{x \in K \mid v(x) > 0\}$.

R is a <u>discrete valuation ring</u> (DVR) if $G = \mathbb{Z}$.

Equivalently, a Noetherian integral domain R is a DVR $\Leftrightarrow$ R is local and not a field and its maximal ideal is principal.

Ex: $R = k[x]_{(x)}$ is a DVR w/ valuation $k(x) - \{0\} \to \mathbb{Z}$ given by $ux^n \mapsto n$ (w/ u a unit in R).

Def: If A, B are local rings contained in a field K, B <u>dominates</u> A if $A \subseteq B$ and $m_B \cap A = m_A$ (m_A and m_B the max'l ideals of A and B, respectively).

Let K be a field, X a scheme. Then a morphism $\operatorname{Spec} K \to X$ is equivalent to a point $x \in X$ and an inclusion $k(x) \subseteq K$.

To see this, let $x \in X$ be the point in the image. Then we get a local homomorphism on stalks

$$\mathcal{O}_{X,x} \longrightarrow K$$

with kernel m_x, so $k(P) = \mathcal{O}_{X,x}/m_x \hookrightarrow K$.

If instead we replace Spec K w/ Spec R, where R is a valuation ring of K, we get the following:

Lemma: Let $T = \operatorname{Spec} R$, with R as above. Then a morphism $T \to X$ is equivalent to a choice of points $x_0, x_1 \in X$ with $x_0 \in \overline{\{x_1\}}$ (i.e. x_1 specializes to x_0, written $x_1 \rightsquigarrow x_0$) and an inclusion of fields $k(x_1) \subseteq K$ such that R dominates $\mathcal{O}_{x_0, Z}$, where Z is the subscheme $\overline{\{x_1\}}$ with its reduced induced structure.

Pf: Let $t_0 = m_R \subseteq R$ the maximal ideal, and $t_1 = (0)$ the generic point.

Consider a morphism $T \to X$, and let x_0 and x_1 be the images of t_0 and t_1. Then the morphism factors through Z:

$$\begin{tikzcd} T \arrow[r] \arrow[dr] & Z \arrow[d] \\ & X \end{tikzcd}$$

Since x_1 is the generic point of Z and Z is reduced, $\mathcal{O}_{x_1, Z}$ is a field (see HW 3), so $k(x_1) = \mathcal{O}_{x_1, Z}$, and it is the field of fractions of $\mathcal{O}_{x_0, Z}$.

So we have local homomorphisms

$$\begin{tikzcd} \mathcal{O}_{x_0,Z} \arrow[r] \arrow[d, hook] & \mathcal{O}_{t_0,\operatorname{Spec} R} = R \arrow[d, hook] \\ k(x_1) \arrow[r, hook] & \mathcal{O}_{t_1,\operatorname{Spec} R} = K \end{tikzcd}$$

and

Thus $\mathcal{O}_{x_0,Z} \subseteq R \subseteq K$, and R dominates $\mathcal{O}_{x_0,Z}$.

Conversely, given x_0, x_1 and an inclusion $k(x_1) \subseteq K$ s.t. R dominates $\mathcal{O}_{x_0,Z}$, we have an inclusion $\mathcal{O}_{x_0,Z} \subseteq R$ which thus induces a morphism

$$T \longrightarrow \operatorname{Spec} \mathcal{O}_{x_0,Z}.$$

Composing this w/ the map $\operatorname{Spec} \mathcal{O}_{x_0,Z} \to X$ we get a morphism $T \to X$. □

We now state the valuative criterion of separatedness. For proof, see Hartshorne.

Noetherian

Thm (Valuative Criterion of Separatedness) Let $f: X \longrightarrow Y$ be a morphism of schemes. f is separated iff the following holds: For any valuation ring R w/ quotient field K, let $T = \operatorname{Spec} R$, $U = \operatorname{Spec} K$, $i: U \to T$. Given morphisms $T \to Y$ and $U \to X$ s.t. the following commutes

$$\begin{tikzcd} U \arrow[r] \arrow[d, "i"'] & X \arrow[d, "f"] \\ T \arrow[r] \arrow[ur, dashed] & Y \end{tikzcd}$$

There is at most one morphism $T \to X$ making the whole diagram commute.

From the theorem, we get several useful corollaries:

Cor: a.) Open + closed immersions are separated.

b.) A composition of separated morphisms is separated.

c.) Separated morphisms are stable under base extension (i.e. if $X \to Y$ is separated and $Y' \to Y$ some morphism, then $X \times_Y Y' \to Y'$ is separated).

d.) $f: X \to Y$ is separated $\iff$ Y has an open cover $\{V_i\}$ s.t. $f^{-1}(V_i) \to V_i$ is separated for each V_i.

e.) If $X \to Y \to Z$ is separated, then $X \to Y$ is separated.

f.) If $X \to Y$ and $X' \to Y'$ are separated, so is $X \times_S X' \to Y \times_S Y'$.

Pf of c.): Suppose $T \to Y'$ and $U \to X \times_Y Y'$ are two morphisms as in the theorem, and suppose we have <u>two</u> maps $T \to X \times_Y Y'$ making the diagram commute:

$$\begin{tikzcd}
U \arrow[r] \arrow[d] & X \times_Y Y' \arrow[r] \arrow[d] & X \arrow[d] \\
T \arrow[r] \arrow[ur, shift left] \arrow[ur, shift right] & Y' \arrow[r] & Y
\end{tikzcd}$$

Then we get two maps $T \to X$. Since $X \to Y$ is separated, these are the same. But then by the univ. property of fiber products, the maps $T \to X \times_Y Y'$ must be the same. $\square$

The proofs of the others are similar.